Taylor's Theorem / Maclaurin's Theorem

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

$$(\text{or}) \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Formula's

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cosh x = x + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\log x = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

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* obtain the Maclaurin's series Expansion of e^x

$$f(x)=e^x \Rightarrow f(0)=e^0=1$$
$$f'(x)=e^x \Rightarrow f'(0)=e^0=1$$
$$f''(x)=e^x \Rightarrow f''(0)=e^0=1$$
$$f'''(x)=e^x \Rightarrow f'''(0)=e^0=1$$

By Maclaurin's Expansion

$$f(x)=f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots$$
$$e^x=1+x(1)+\frac{x^2}{2!}(1)+\frac{x^3}{3!}(1)+\cdots$$
$$\boxed{e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}$$

* $f(x)=\sin x$

Sol

$$f(x)=\sin x \Rightarrow f(0)=\sin 0=0$$
$$f'(x)=\cos x \Rightarrow f'(0)=\cos 0=1$$
$$f''(x)=-\sin x \Rightarrow f''(0)=-\sin 0=0$$
$$f'''(x)=-\cos x \Rightarrow f'''(0)=-\cos 0=-1$$
$$f^{IV}(x)=+\sin x \Rightarrow f^{IV}(0)=\sin 0=0$$
$$f^{V}(x)=\cos x \Rightarrow f^{V}(0)=\cos 0=1$$

By Maclaurin's Expansion

$$f(x)=f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{IV}(0)+\frac{x^5}{5!}f^{V}(0)$$
$$\sin x=0+x(1)+\frac{x^2}{2!}(0)+\frac{x^3}{3!}(-1)+\frac{x^4}{4!}(0)+\frac{x^5}{5!}(1)$$
$$\boxed{\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots}$$

(iii) $f(x) = \cos x$

Sol $f(x) = \cos x \Rightarrow f(0) = \cos 0 = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = \sin 0 = 0$

$f^{IV}(x) = \cos x \Rightarrow f^{IV}(0) = \cos 0 = 1$

By Maclaurin's Expansion

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \cdots$$

$$\cos x = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

(iv) $f(x) = \log(1+x)$

$f(x) = \log(1+x) \Rightarrow f(0) = \log(1+0) = \log 1 = 0$

$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = \frac{1}{1+0} = 1$

$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = \frac{-1}{(1+0)^2} = -1$

$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = \frac{2}{(1+0)^3} = 2$

$f^{IV}(x) = \frac{-6}{(1+x)^4} \Rightarrow f^{IV}(0) = \frac{-6}{(1+0)^4} = -6$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \cdots$$

$$\log(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

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* Express the polynomail $2x^3+7x^2+x-6$ in power of $x-2$ using taylor's series.

$x-2=0 \Rightarrow x=2$

Sol

$f(x) = 2x^3+7x^2+x-6 \Rightarrow f(2) = 40$

$f'(x) = 6x^2+14x+1 \Rightarrow f'(2) = 53$

$f''(x) = 12x+14 \Rightarrow f''(2) = 38$

$f'''(x) = 12 \Rightarrow f'''(2) = 12$

$f^{IV}(x) = 0 \Rightarrow f^{IV}(2) = 0$

By taylor's series

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

$$f(x) = f(2) + (x-2) f'(2) + \frac{(x-2)^2}{2} f''(2) + \frac{(x-2)^3}{3!} f'''(2) + \cdots$$

$$f(x) = 40 + (x-2)(53) + \frac{(x-2)^2}{2}(38) + \frac{(x-2)^3}{6}(12)$$

$$f(x) = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3 + \cdots$$

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Obtain the taylor's series Expansion of $\sin x$ in powers of $x-\frac{\pi}{4}$

$x-\frac{\pi}{4}=0 \quad x=\frac{\pi}{4}$

Sol

$$f(x)=\sin x \Rightarrow f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

$$f'(x)=\cos x \Rightarrow f'\left(\frac{\pi}{4}\right)=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

$$f''(x)=-\sin x \Rightarrow f''\left(\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=\frac{-1}{\sqrt{2}}$$

$$f'''(x)=-\cos x \Rightarrow f'''\left(\frac{\pi}{4}\right)=-\cos\frac{\pi}{4}=\frac{-1}{\sqrt{2}}$$

By taylor's series

$$f(x)=f(a)+\frac{(x-a)^1}{1!}f'(a)+\frac{}{2!}f''(a)+\frac{}{3!}f'''(a)+\cdots$$

$$\sin x=f\left(\frac{\pi}{4}\right)+\left(x-\frac{\pi}{4}\right)f'\left(\frac{\pi}{4}\right)+\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right)+\frac{\left(x-\frac{\pi}{4}\right)^3}{3!}f'''\left(\frac{\pi}{4}\right)+\cdots$$

$$\sin x=\frac{1}{\sqrt{2}}+\left(x-\frac{\pi}{4}\right)\frac{1}{\sqrt{2}}+\frac{\left(x-\frac{\pi}{4}\right)^2}{2}\left(-\frac{1}{\sqrt{2}}\right)+\frac{\left(x-\frac{\pi}{4}\right)^3}{6}\left(-\frac{1}{\sqrt{2}}\right)+\cdots$$

$$\sin x=\frac{1}{\sqrt{2}}\left[1+\left(x-\frac{\pi}{4}\right)-\frac{\left(x-\frac{\pi}{4}\right)^2}{2}-\frac{\left(x-\frac{\pi}{4}\right)^3}{6}\cdots\right]$$

* obtain Taylor's series expansion of e^{x} about $x=-1$

$$f(x)=e^{x} \Rightarrow f(-1)=e^{-1}=1/e$$
$$f'(x)=e^{x} \Rightarrow f'(-1)=e^{-1}=1/e$$
$$f''(x)=e^{x} \Rightarrow f''(-1)=e^{-1}=1/e$$
$$f'''(x)=e^{x} \Rightarrow f'''(-1)=e^{-1}=1/e$$

By taylor's Theorem

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\cdots$$

$$e^{x}=f(-1)+(x+1)f'(-1)+\frac{(x+1)^2}{2}f''(-1)+\frac{(x+1)^3}{3!}f'''(-1)+\cdots$$

$$e^{x}=\frac{1}{e}+(x+1)\frac{1}{e}+\frac{(x+1)^2}{2!}\frac{1}{e}+\frac{(x+1)^3}{3!}\,1/e+\cdots$$

$$e^{x}=\frac{1}{e}\left[1+(x+1)+\frac{(x+1)^2}{2!}+\frac{(x+1)^3}{3!}+\cdots\right]$$